# Logic in Computer Science

## Lecture 03
## Predicate Logic

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# Predicate Logic (1)

- ★ The need for a richer language
- ★ Predicate logic as a formal language
  - – terms —variables, functions
  - – formulas —predicates, quantifiers
  - – free and bound variables
  - – substitution
- ★ Proof theory of predicate logic
  - – Natural deduction rules

## The Need for a Richer Language

**Propositional Logic:**

- Study of declarative sentences, statements about the world which can be given a truth value
- Dealt very well with sentence components like: ***not***, ***and***, ***or***, ***if*** $\cdots$ ***then*** $\cdots$
- **Limitations:** cannot deal with modifiers like ***there exists***, ***all***, ***among***, ***only***.

**Example:** ***"Every student is younger than some instructor."***

- We could identify the entire phrase with the propositional symbol $p$.
- However, the phrase has a finer logical structure. It is a statement about the following properties:
  - being a student
  - being an instructor
  - being younger than somebody else

# Predicates, Variables and Quantifiers

Properties are expressed by predicates. $S$, $I$, $Y$ are ***predicates***.

$S(andy)$: Andy is a ***student***.
$I(paul)$: Paul is an ***instructor***.
$Y(andy, paul)$: Andy is ***younger than*** Paul.

Variables are placeholders for concrete values.

$S(x)$: $x$ is a student.
$I(x)$: $x$ is an instructor.
$Y(x, y)$: $x$ is younger than $y$.

Quantifiers make possible encoding the phrase:

***"Every student is younger than some instructor."***

Two quantifiers: $\forall$ —***forall***, and $\exists$ —***exists***.

Encoding of the above sentence:

$$\forall x(S(x) \rightarrow (\exists y(I(y) \wedge Y(x, y))))$$

## More Examples

***"No books are gaseous. Dictionaries are books. Therefore, no dictionary is gaseous."***

We denote: $B(x)$: $x$ is a book
$G(x)$: $x$ is gaseous
$D(x)$: $x$ is a dictionary

$$\neg\exists x\,(B(x) \land G(x)), \forall x\,(D(x) \to B(x))$$
$$\vdash$$
$$\neg\exists x\,(D(x) \land G(x))$$

***"Every child is younger than his mother"***

We denote: $C(x)$: $x$ is a child
$M(x,y)$ : $x$'s mother is $y$

$$\forall x \forall y\,(C(x) \land M(x,y) \to Y(x,y))$$

Denote $m(x)$: mother of $x$

$$\forall x\,(C(x) \to Y(x, m(x)))$$

Using the function $m$ to encode the "mother of" relationship is more appropriate, since every person has a unique mother.

## More Examples (2)

***"Andy and Paul have the same maternal grandmother"***

$$\forall x \forall y \forall u \forall v (M(a,x) \wedge M(x,y) \wedge M(p,u) \wedge M(u,v) \rightarrow y = v)$$

We have introduced a new, special predicate: ***equality***.

Alternative representation:

$$m(m(a)) = m(m(p))$$

Consider the relationship $B(x,y)$: $x$ is the brother of $y$. This relationship must be encoded as a predicate, since a person may have more than one brother.

# Predicate Logic as a Formal Language

Two sorts of "things" in a predicate formula:

- Objects such as $a$ (Andy) and $p$ (Paul). Function symbols also refer to objects. These are modeled by ***terms***.
- Expressions that can be given truth values. These are modeled by ***formulas***.

A predicate vocabulary consists of 3 sets:

- ***Predicate symbols*** $\mathcal{P}$;
- ***Function symbols*** $\mathcal{F}$;
- ***Constants*** $\mathcal{C}$.

Each predicate and function symbol comes with a fixed ***arity*** (i.e. number of arguments)

Elements of the formal language of predicate logic:

- Terms
- Formulas
- Free and bound variables
- Substitution

# Terms

**Definition:** ***Terms*** are defined as follows:

- Any variable is a term;
- Any constant in $\mathcal{C}$ is a term;
- If $t_1,\ldots,t_n$ are terms and $f \in \mathcal{F}$ has arity $n$, then $f(t_1,\ldots,t_n)$ is a term;
- Nothing else is a term.

**Backus-Naur definition:** $t ::= x \,|\, c \,|\, f(t,\ldots,t)$ where $x$ represents variables, $c$ represents constants in $\mathcal{C}$, and $f$ represents function in $\mathcal{F}$ with arity $n$.

**Remarks:**

- The first building blocks of terms are constants and variables.

- More complex terms are built from function symbols using previously buit terms.

- The notion of terms is independent on the sets $\mathcal{C}$ and $\mathcal{F}$.

# Formulas

**Definition:** We define the set of ***formulas*** over $(\mathcal{F}, \mathcal{P})$ inductively, using the already defined set of terms over $\mathcal{F}$.

- If $P$ is a predicate with $n \geq 1$ arguments, and $t_1,\ldots,t_n$ are terms over $\mathcal{F}$, then $P(t_1,\ldots,t_n)$ is a formula.
- If $\Phi$ is a formula, then so is $\neg\Phi$.
- If $\Phi$ and $\Psi$ are formulas, then so are $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \rightarrow \Psi$.
- If $\Phi$ is a formula and $x$ is a variable, then $\forall x \Phi$ and $\exists x \Phi$ are formulas.
- Nothing else is a formula.

**BNF definition:**

$$\Phi ::= P(t_1,\ldots t_n) \,|\, (\neg\Phi) \,|\, (\Phi \wedge \Phi) \,|\, (\Phi \vee \Phi) \,|\, (\Phi \rightarrow \Phi) \,|\, (\forall x \Phi) \,|\, (\exists x \Phi)$$

where $P$ is a predicate of arity $n$, $t_i$ are terms, $i \in \{1,\ldots,n\}$, $x$ is a variable.

**Convention:** We retain the usual binding priorities of the connectives $\neg$,$\wedge$, $\vee$, $\rightarrow$. We add that $\forall x$ and $\exists x$ bind like $\neg$.

# Example

Consider translating the sentence:

***"Every son of my father is my brother"***

Two alternatives:

- ***"Father of"*** relationship encoded as a predicate.

  $S(x,y)$: $x$ is the son of $y$.

  $F(x,y)$: $x$ is the father of $y$.

  $B(x,y)$: $x$ is the brother of $y$.

  $m$: constant, denoting "myself".

  **Translation:** $\forall x \forall y (F(x,m) \wedge S(y,x) \rightarrow B(y,m))$

- ***"Father of"*** relationship encoded as a function.

  $f(x)$: father of $x$.

  **Translation:** $\forall x (S(x, f(m)) \rightarrow B(x,m))$

## Free and Bound Variables

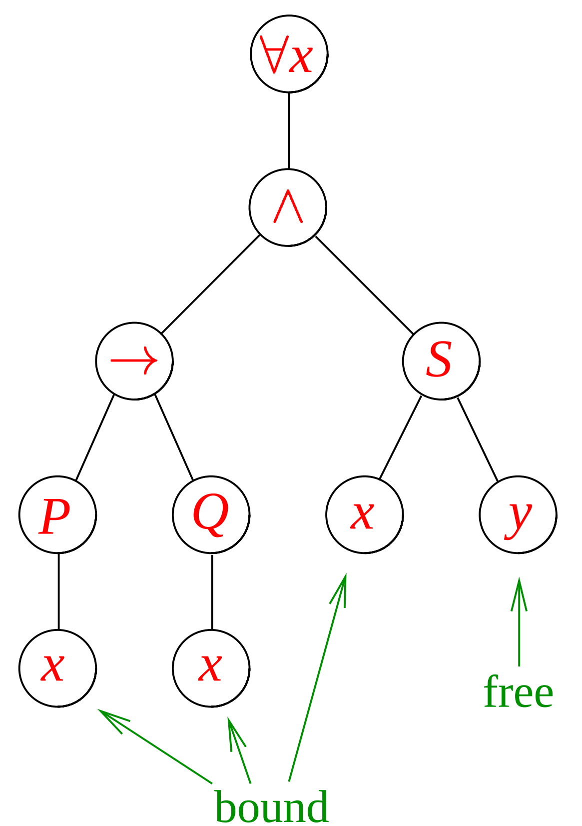
**Definition:** Let $\Phi$ be a formula in predicate logic. An occurrence of $x$ in $\Phi$ is ***free in*** $\Phi$ if it is a leaf node in the parse tree of $\Phi$ such that there is no path upwards from that node $x$ to a node $\forall x$ or $\exists x$. Otherwise, that occurrence $x$ is called ***bound***. For $\forall x \Phi$, we say that $\Phi$ —minus any of its sub-formulas $\exists x \Psi$, or $\forall x \Psi$ —is the scope of $\forall x$, respectively $\exists x$.

Formula:

$$\forall x \underbrace{((P(x) \to Q(x)) \land S(x,y))}_{\text{Scope of } \forall x}$$

$x$ is bound.

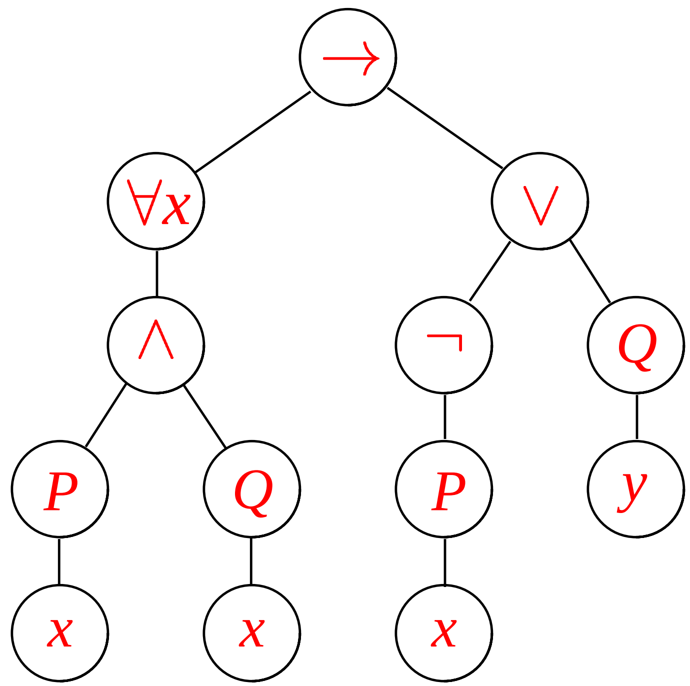
$y$ is free.



# Examples of Free and Bound Variables

Formula: $(\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y))$

Parse tree:

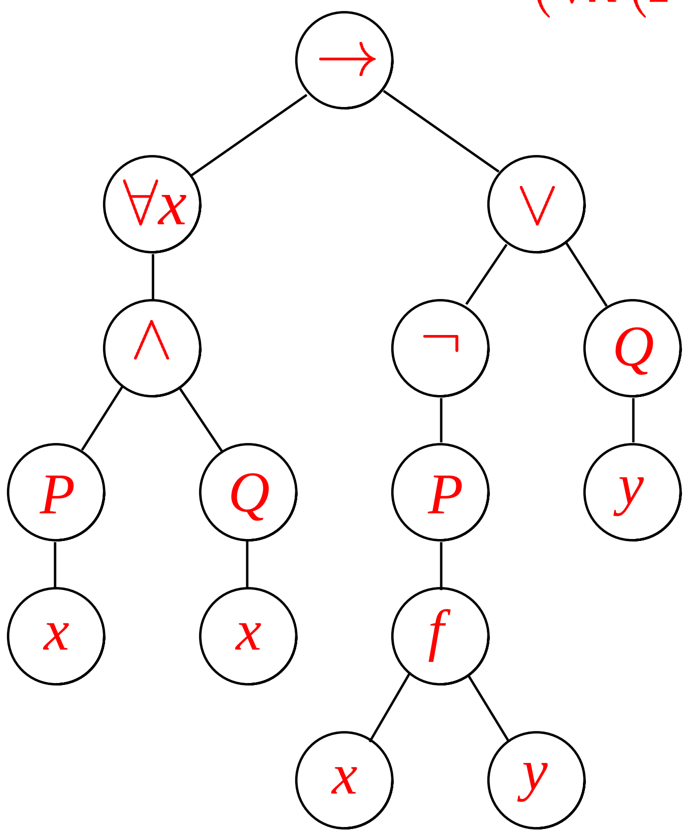


Variables are placeholders, so we must have means of ***replacing*** them with more concrete information.

**Definition:** Given a variable $x$, a term $t$, and a formula $\Phi$, we define $\Phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable $x$ in $\Phi$ with $t$.

$((\forall x\,(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))[f(x,y)/x]$ is

$$(\forall x\,(P(x) \wedge Q(x))) \rightarrow (\neg P(f(x,y)) \vee Q(y))$$



## Substitution (2)

**Definition:** Given a term $t$, a variable $x$, and a formula $\Phi$, we say that $t$ ***is free for*** $x$ ***in*** $\Phi$ if no free $x$ leaf in $\Phi$ occurs in the scope of $\forall y$ or $\exists y$, for every variable $y$ occurring in $t$.

**Remark:** If $t$ is not free for $x$ in $\Phi$, then the substitution $\Phi[t/x]$ has unwanted effects.

**Example:**

$$(S(x) \wedge (\forall y\,(P(x) \to Q(y))))[y/x] \text{ is } S(y) \wedge (\forall y\,(P(y) \to Q(y)))$$

Avoid this by renaming $\forall y$ into $\forall z$.

$$(S(x) \wedge (\forall z\,(P(x) \to Q(z))))[y/x] \text{ is } S(y) \wedge (\forall z\,(P(y) \to Q(z)))$$

- Natural deduction rules for propositional logic are still valid
- Natural deduction rules for predicate logic:
  - proof rules from propositional logic;
  - proof rules for equality;
  - proof rules for universal quantification;
  - proof rules for existential quantification.
- Quantifier equivalences

# Proof Rules for Equality

$$\frac{}{t = t} \; = \mathrm{i} \qquad\qquad \frac{t_1 = t_2 \quad \Phi[t_1/x]}{\Phi[t_2/x]} \; = \mathrm{e}$$

**Convention:** When we write a substitution in the form $\Phi[t/x]$, we implicitly assume that $t$ is free for $x$ in $\Phi$.

**Proof example:**

$$x+1 = 1+x, (x+1>1) \to (x+1>0) \vdash (1+x>1) \to (1+x>0)$$

| | | |
|---|---|---|
| 1 | $x+1 = 1+x$ | premise |
| 2 | $(x+1>1) \to (x+1>0)$ | premise |
| 3 | $(1+x>1) \to (1+x>0)$ | $=$e 1,2 |

# Proof Rules for Universal Quantification

$$\frac{\forall x\, \Phi}{\Phi[t/x]}\ \forall x\, \mathrm{e}$$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad \Phi[x_0/x] \end{array}}}{\forall x\, \Phi}\ \forall x\, \mathrm{i}$$

**Proof examples:**

$\forall x\,(P(x) \to Q(x)), \forall x\, P(x) \vdash \forall x\, Q(x)$

| | | | |
|---|---|---|---|
| 1 | | $\forall x\,(P(x) \to Q(x))$ | premise |
| 2 | | $\forall x\, P(x)$ | premise |
| 3 | $x_0$ | $P(x_0) \to Q(x_0)$ | $\forall x$ e 1 |
| 4 | | $P(x_0)$ | $\forall x$ e 2 |
| 5 | | $Q(x_0)$ | $\to$e 3,4 |
| 6 | | $\forall x\, Q(x)$ | $\forall x$ i 3-5 |

$P(t), \forall x\,(P(x) \to \neg Q(x)) \vdash \neg Q(t)$

| | | |
|---|---|---|
| 1 | $P(t)$ | premise |
| 2 | $\forall x\,(P(x) \to \neg Q(x))$ | premise |
| 3 | $p(t) \to \neg Q(t)$ | $\forall x$ e 2 |
| 4 | $\neg Q(t)$ | $\to$ e 3,1 |

# Proof Rules for Existential Quantification

$$\frac{\Phi[t/x]}{\exists x\,\Phi}\ \exists x\,\mathrm{i}$$

$$\frac{\exists x\,\Phi \qquad \boxed{\begin{array}{ll} x_0 & \Phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists x\,\mathrm{e}$$

Side condition: $x_0$ doesn't occur in $\chi$

**Proof examples:**

$\forall x\,\Phi \vdash \exists x\,\Phi$

| | | |
|---|---|---|
| 1 | $\forall x\,\Phi$ | premise |
| 2 | $\Phi[x/x]$ | $\forall x$ e 1 |
| 3 | $\exists x\,\Phi$ | $\exists x$ i 2 |

$\forall x\,(P(x) \rightarrow Q(x)), \exists x\,P(x) \vdash \exists x\,Q(x)$

| | | | |
|---|---|---|---|
| 1 | | $\forall x\,(P(x) \rightarrow Q(x))$ | premise |
| 2 | | $\exists P(x)$ | premise |
| 3 | $x_0$ | $P(x_0)$ | assumption |
| 4 | | $P(x_0) \rightarrow Q(x_0)$ | $\forall x$ e 1 |
| 5 | | $Q(x_0)$ | $\rightarrow$ e 4,3 |
| 6 | | $\exists x\,Q(x)$ | $\exists x$ i 5 |
| 7 | | $\exists x\,Q(x)$ | $\exists x$ e 2,3–6 |

# Another Example

$$\exists x\, P(x), \forall x \forall y\, (P(x) \rightarrow Q(y)) \vdash \forall y\, Q(y)$$

| | | | |
|---|---|---|---|
| 1 | | $\exists x\, P(x)$ | premise |
| 2 | | $\forall x \forall y\, (P(x) \rightarrow Q(y))$ | premise |
| 3 | $y_0$ | | |
| 4 | $x_0$ | $P(x_0)$ | assumption |
| 5 | | $\forall y\, (P(x_0) \rightarrow Q(y))$ | $\forall x$ e 2 |
| 6 | | $P(x_0) \rightarrow Q(y_0)$ | $\forall y$ e 2 |
| 7 | | $Q(y_0)$ | $\rightarrow$ 6,4 |
| 8 | | $Q(y_0)$ | $\exists x$ e 1,4-7 |
| 9 | | $\forall y\, Q(y)$ | $\forall y$ i 3–8 |